

# Fibonomial Cumulative Connection Constants

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## Abstract

In this note we present examples of cumulative connection constants - new Fibonomial ones included. All examples posses combinatorial interpretation.

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## 1 Introduction

The cumulative connection constants (**ccc**)  $C_n$  were introduced by the Authors of [1]:

$$C_n = \sum_{k \geq 0} c_{n,k}, \quad n \geq 0,$$

where

$$p_n(x) = \sum_{0 \leq k \leq n} c_{n,k} q_k(x)$$

and Pascal-like array  $\{c_{n,k}\}_{n,k \geq 0}$  of connection constants  $c_{n,k}$ ,  $n, k \geq 0$  "connects" two polynomial sequences (Note:  $\{p_k(x)\}_{k \geq 0}$  is a polynomial sequence if  $\deg p_k = k$ ).

#### Motivation 1

"The connection constants problem" i.e. combinatorial interpretations, algorithms of calculation , recurrences and other properties - is one of the central issues of the binomial enumeration in Finite Operator Calculus (FOC ) of Rota-Roman and Others and in its afterwards extensions (see: abundant references in [8]-[10],[13]). Important knowledge on recurrences for  $\{c_{n,k}\}_{n,k \geq 0}$  one draws from [2] - consult also the NAVIMA group program:  
<http://webs.uvigo.es/t10/navima/> .

#### Motivation 2

The cumulative connection constants (**ccc**) as defined above appear to represent combinatorial quantities of primary importance - as shown by examples below .

A special case of interest considered in [2, 1] is the case of monic persistent root polynomial sequences [3] characterized by the following conditions:

$$p_0(x) = q_0(x) = 1 ; \quad q_n(x) = q_{n-1}(x)(x-r_n) , \quad p_n(x) = p_{n-1}(x)(x-s_n) \quad n \geq 1.$$

The pair of persistent root polynomial sequences and hence corresponding connection constants array are then bijectively labeled by the pair of root sequences :

$$[r] = \{r_k\}_{k \geq 1} , \quad [s] = \{s_k\}_{k \geq 1} .$$

In this particular case of persistent root polynomial sequences the authors of [2] gave a particular name to connection constants : "generalized Lah numbers" - now denoted as follows  $c_{n,k} = L_{n,k}$ . The generalized Lah numbers do satisfy the following recurrence [2, 1] :

$$L_{n+1,k} = L_{n,k-1} + (r_{k+1} - s_{n+1})L_{n,k} \tag{1}$$

$$L_{0,0} = 1 , \quad L_{0,-1} = 0 \quad n, k \geq 0$$

from which the recurrence for cumulative connection constants (**ccc**) follows

$$C_{n+1} = (1 - s_{n+1})C_n + \sum_{k=0}^n c_{n,k}r_{k+1} \quad (2)$$

where  $c_{n,k} = L_{n,k}$ .

The clue examples of [1] are given by the following choice of a pair of root sequences :

the Fibonacci sequence

$$C_n = F_n; \quad n > 0 \longleftrightarrow [r] = \{r_k\}_{k \geq 1} = \{0, 0, 0, 0, 0, 0, \dots\}, \quad [s] = \{s_k\}_{k \geq 1} = \{0, 0, \dots, 0, \dots\} = [0]$$

the Lucas sequence

$$C_n = L_n; \quad n > 0 \longleftrightarrow [r] = \{r_k\}_{k \geq 1}; \quad r_1 = 0, r_2 = 2, r_k = 1 - (r_{k-1})^2 \text{ for } k > 2 \text{ and } [s] = [0].$$

## 2 Examples of cumulative connection constants - old and new

We shall supply now several examples of connection constants arrays and corresponding ccc's - Fibonomial case included - altogether with their combinatorial interpretation. (HELP! - For notation-help - see: Appendix ).

**Example 1.**  $c_{n,k} = \binom{n}{k}$ ,  $x^n = \sum_{k \geq 0} \binom{n}{k} (x-1)^k \quad n \geq 0$ ;  
 $E^{-1}(D)x^k = (x-1)^k \quad n \geq 0$ .

$$C_n = \sum_{k \geq 0} \binom{n}{k} = 2^n = \text{number of all subsets of } S; \quad n = |S|; \quad n \geq 0.$$

Here  $E^a(D) = \sum_{n \geq 0} \frac{a^n}{n!} D^n$  is the translation operator ( $D = \frac{d}{dx}$ ) and the recurrence (2) is obvious.

**Example 2.**  $c_{n,k} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  denote Stirling numbers of the second type. Then  $x^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k$ . Let  $B_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ , then  $B_n = \text{number of all partitions of } S; n = |S|; B_n \equiv \text{Bell numbers}$ . As the recurrence for Bell numbers reads  $B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k$ ;  $n \geq 0$  we easily get from (2) the inspiring identity

$$\sum_{k=0}^n \binom{n}{k} B_k = B_n + \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k.$$

**Example 3.**  $c_{n,k} = \left[ \begin{matrix} n \\ k \end{matrix} \right]$  are Stirling numbers of the first kind. Then  $x^{\bar{n}} = \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] x^k$ ,  $n \geq 0$ .  $C_n = \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] = n!$  and the obvious recurrence  $C_{n+1} = (n+1)C_n$  coincides with (2).

**Example 4.**  $c_{n,k} = \binom{n}{k}_q$ ,  $x^n = \sum_{k \geq 0} \binom{n}{k}_q \Phi_k(x)$ ,  $\Phi_k(x) = \prod_{s=0}^{k-1} (x - q^s)$ ,

$$C_n = \sum_{k \geq 0} \binom{n}{k}_q = \text{number of all subspaces of } V(n, q) \text{ i.e.}$$

$$C_n = \sum_{k=0}^n \binom{n}{k}_q = |L(n, q)| \equiv (1 +_q 1)^n,$$

where  $L(n, q)$  denotes the lattice of all subspaces of  $V(n, q)$  - (the  $n$ -th dimensional space  $V(n, q)$  over Galois field  $GF(q)$ ) - see: [4]. According to Konvalina

$$C_n = \sum_{k=0}^n \binom{n}{k}_q \equiv (1 +_q 1)^n$$

is at the same time the number of all possible choices with repetition of objects from exponentially weighted boxes [5, 6].

As  $q$ -Gaussian polynomials  $\Phi_k(x) = \prod_{s=0}^{k-1} (x - q^s)$  constitute a persistent root polynomial sequence - the generalized Lah numbers - which are now are  $\binom{n}{k}_q = c_{n,k} = L_{n,k}$  must satisfy the recurrence equation

$$L_{n+1,k} = L_{n,k-1} + r_{k+1} L_{n,k}, \quad L_{0,0} = 1, \quad L_{0,-1} = 0 \quad n, k \geq 0,$$

where  $[r] = \{q^{k-1}\}_{k \geq 1}$  is the root sequence determining  $\{\Phi_n\}_{n \geq 0}$ . See Appendix 2. Hence the recurrence (2) for the number  $C_n$  of all possible choices

with repetition of objects from exponentially weighted boxes [5, 6] is of the form

$$C_{n+1} = C_n + \sum_{k=0}^n \binom{n}{k}_q q^k$$

because

$$C_{n+1} = C_n + \sum_{k=0}^n c_{n,k} r_{k+1}.$$

In another  $q$ -umbral notation (see: Appendix 2)

$$(1 +_q 1)^{n+1} = (1 +_q 1)^n + \sum_{k=0}^n \binom{n}{k}_q q^k.$$

Occasionally note that using the Möbius inversion formula (see: [4] - the  $q$ -binomial theorem Corollaries in section 5 - included ) we have

$$x^n = \sum_{k \geq 0} \binom{n}{k}_q \Phi_k(x), \quad \Phi_k(x) = \sum_{l \geq 0} \binom{k}{l}_q (-1)^l q^{\binom{l}{2}} x^{k-l}, \quad n \geq 0$$

i.e. the identity

$$\prod_{s=0}^{k-1} (x - q^s) \equiv \sum_{l \geq 0} \binom{k}{l}_q {}_1^l q^{\binom{l}{2}} x^{k-l}.$$

"The Möbius  $q$ -inverse example" is then the following.

**Example 4'.**  $c_{n,k} = \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}}$ ,  $x^n = \sum_{k \geq 0} c_{n,k} H_k(x)$ , where (written with help of generalized shift operator [7, 8, 9, 10] - analogously to the use of  $E^{-1}(D)x^k = (x - 1)^k$  above in Example 1. ) we have (see Appendix 2):

$$H_k(x) = E(\partial_q) x^k \equiv (x +_q 1)^k = \sum_{l \geq 0} \binom{k}{l}_q x^l.$$

Compare with  $H_k(x, y)|_{y=1} = H_k(x)$  in [11] (pp. 240-241) and note that

$$C_n = \sum_{k=0}^n |c_{n,k}| = \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}}$$

is the number of all possible choices without repetition of objects from exponentially weighted boxes [5, 6]. It is then combinatorial - natural to consider also another triad .

**Example 4”.**  $c_{n,k} = \binom{n}{k}_q q^{\binom{k}{2}}$ ,  $x^n = \sum_{k \geq 0} \binom{n}{k}_q q^{\binom{k}{2}} \Gamma_k(x)$ ,  
 $\Gamma_k(x) = \sum_{l \geq 0} \binom{k}{l}_q (-1)^{k-l} x^l$ ,  $n \geq 0$ .

**Example 5.** This is The Fibonomial Example:  $c_{n,k} = \binom{n}{k}_F$ ,  
 $x^n = \sum_{k \geq 0} \binom{n}{k}_F \Xi_k(x)$  where  $\binom{n}{k}_F$  are called Fibonomial coefficients (see A3).  
Using the Möbius inversion formula - analogously to the Example 4 case - we get

$$\Xi_k(x) = \sum_{l \geq 0} \binom{k}{l}_F \mu(l, k) x^{k-1}, \quad n \geq 0,$$

where Möbius matrix  $\mu$  is the unique inverse of the incidence matrix  $\zeta$  defined for the "Fibonacci cobweb" poset introduced in [12] (see: A4 for explicit formula of  $\zeta$  from [16]). For combinatorial interpretation of these Fibonomial coefficients see: Appendix 4. Interpretation of  $C_n$  stated below results from combinatorial interpretation of Fibonomial coefficients. Combinatorial interpretation of ccc  $C_n$  in this case is then the following:

$C_n = \sum_{k \geq 0} \binom{n}{k}_F$  = number of all such subposets of the Fibonacci cobweb poset  $P$  which are cobweb subposets starting from the level  $F_k$  and ending at the level labeled by  $F_n$ .

As in the  $q$ -umbral case of the Example 4 we have "F-umbral" representation of the Fibonomial ccc  $C_n$  ( see A3) . Namely

$$C_n = \sum_{k=0}^n \binom{n}{k}_F \equiv (1 +_F 1)^n$$

### 3 Remark on ccc - recurrences "exercise"

ccc-recurrences from examples 1-4 were an easy game to play. All these four cases of ccc might be interpreted on equal footing due to the ingenious scheme of one unified combinatorial interpretation by Konvalina [5, 6].

As for the other cases the "exercise" of finding the recurrences for ccc - up to the Fibonomial case - might be the slightly harder task. In this last case with Fibonomial coefficients one may derive the well known recurrence

$$\begin{aligned} \binom{n+1}{k}_F &= F_{k-1} \binom{n}{k}_F + F_{n-k+2} \binom{n}{k-1}_F \\ \binom{n}{0}_F &= 1, \quad \binom{0}{k}_F = 0 \text{ for } k > 0 \end{aligned} \tag{3}$$

or equivalently

$$\begin{aligned}\binom{n+1}{k}_F &= F_{k+1} \binom{n}{k}_F + F_{n-k} \binom{n}{k-1}_F \\ \binom{n}{0}_F &= 1, \quad \binom{0}{k}_F = 0 \quad \text{for } k > 0\end{aligned}$$

also by combinatorial reasoning [16] (it might be a hard exercise - not to check it but to prove it another way - a use of [2] - being recommended). Derivation of the recurrence for Fibonomial ccc  $C_n$  - we leave as an exercise.

## 4 Appendix

### 4.1 On the notation used through out in this note

$q$ -binomial and Fibonomial cases are specifications of the  $\psi$ -sequence choice in the so called  $\psi$ -calculus notation [8]-[10] in conformity with [14] referring back to works of Brenke and Boas on generalized Appell polynomials (see abundant references in [9])

### 4.2 $q$ -Gaussian ccc

$$\begin{aligned}\binom{n+1}{k}_q &= q^k \binom{n}{k}_q + \binom{n}{k-1}_q \\ \binom{n}{0}_q &= 1, \quad n \geq 0, k \geq 1,\end{aligned}\tag{4}$$

where  $n_q! = n_q(n-1)_q!;$   $1_q! =_q 1 = 1,$   $n_q^k = n_q(n-1)_q \dots (n-k+1)_q,$   $\binom{n}{k}_q \equiv \frac{n_q^k}{k_q!}.$

**Remark 4.1.** (see [15]) The dual to (4) recurrence is then given by

$$x\Phi_n(x) = q^n\Phi_n(x) + \Phi_{n+1}(x)\tag{5}$$

$$\Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad n \geq 0,$$

in accordance with a well known fact (see: [4] ) that

$$x^n = \sum_{k \geq 0} \binom{n}{k}_q \Phi_k(x),\tag{6}$$

where  $\Phi_k(x) = \prod_{s=0}^{k-1}(x - q^s)$  are the  $q$ -Gaussian polynomials.

Recall now that the Jackson  $\partial_q$ - derivative is defined as follows [7]-[11]:  $(\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}$  - and is also called the  $q$ -derivative. The consequent notation for the generalized shift operator [7]-[10],[13] is:

$$E^a(\partial_q)x^n = \exp_q\{a\partial_q\} = \sum_{k=0}^{\infty} \frac{a^k}{k_q!} \partial_q^k.$$

Then we identify  $E^y(\partial_q)x^n \equiv (x +_q y)^n$ . (Ward in [7] does not use any subscript like  $q$  or  $F$  or  $\psi$ .

Naturally his "+" means not + but a "plus with a subscript" in our notation).

### 4.3 Fibonomial notation

In straightforward analogy consider now the *Fibonomial coefficients*  $c_{n,k}$ ,

$$c_{n,k} = \binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!} = \binom{n}{n-k}_F,$$

where  $n_F \equiv F_n, n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \dots 2_F 1_F; 0_F! = 1; n_F^k = n_F(n-1)_F \dots (n-k+1)_F; \binom{n}{k}_F \equiv \frac{n_F^k}{k_F!}$  and linear difference operator  $\partial_F$  acting as follows:  $\partial_F; \partial_F x^n = n_F x^{n-1}; n \geq 0$  - we shall call the  $F$ -derivative. Then in conformity with [7] and with notation as in [10, 9, 8, 13] one writes:

$$(1) \quad (x +_F a)^n \equiv \sum_{k \geq 0} \binom{n}{k}_F a^k x^{n-k} \text{ where } \binom{n}{k}_F \equiv \frac{n_F^k}{k_F!}$$

and  $n_F^k = n_F(n-1)_F \dots (n-k+1)_F;$

$$(2) \quad (x +_F a)^n \equiv E^a(\partial_F x^n); \quad E^a(\partial_F) = \sum_{n \geq 0} \frac{a^n}{n_F!} \partial_F^n;$$

$E^a(\partial_F)f(x) = f(x +_F a), E^a(\partial_F)$  is corresponding generalized translation operator.

### 4.4 Combinatorial interpretation of Fibonomial Coef-ficients [12, 16, 17, 18, 19, 20]

Let us depict a partially ordered infinite set  $P$  via its finite part Hasse diagram to be continued ad infinitum in an obvious way as seen from the figure below.

This Fibonacci cobweb partially ordered infinite set  $P$  is defined as in [12] via its finite part - cobweb subposet  $P_m$ (rooted at  $F_1$  level subposet) to be continued ad infinitum in an obvious way as seen from the figure of  $P_5$  below. It looks like the Fibonacci tree with a specific "cobweb". It is identified with Hasse diagram of the partial order set  $P$ .

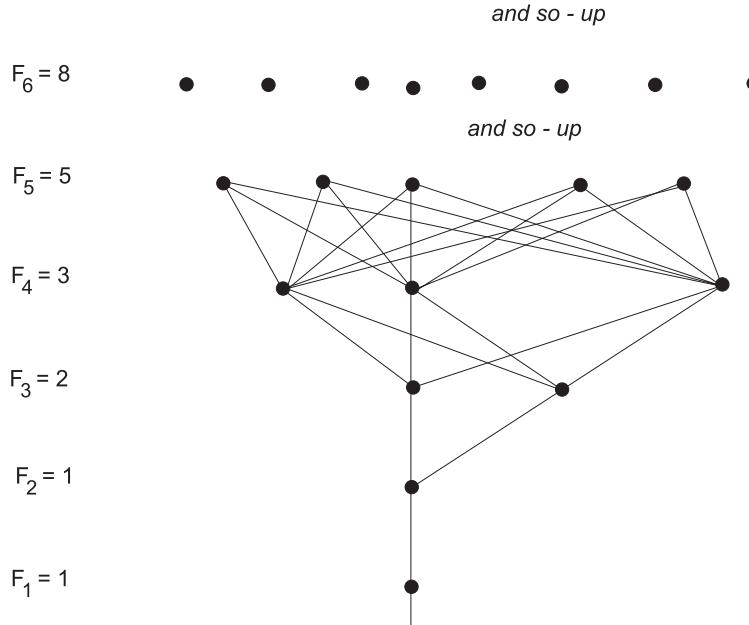


Figure 1: The graded structure of the Fibonacci cobweb poset

If one defines this poset  $P$  with help of its incidence matrix  $\zeta$  representing  $P$  uniquely then one arrives at  $\zeta$  with easily recognizable staircase-like structure - of zeros in the upper part of this upper triangle matrix  $\zeta$ . This structure is depicted by the Figure 2 where: empty places mean zero values (under diagonal) and in filled with - - - parts of rows (above the diagonal) " - " stay for ones. The picture below is drawn for the sequence  $F = \langle F_1, F_2, F_3, \dots, F_n, \dots \rangle$ , where  $F_k$  are Fibonacci numbers.

	1	- - -
	1 0	- - -
	1	- - -
	1 0 0	- - -
	1 0	- - -
	1	- - -
$F_5 - 1 \text{ } 0's$	1 0 0 0 0	- - -
	1 0 0 0	- - -
	1 0 0	- - -
	1 0	- - -
	1	- - -
$F_6 - 1 \text{ zeros}$	1 0 0 0 0 0 0	- - -
	1 0 0 0 0 0	- - -
	1 0 0 0 0	- - -
	1 0 0 0	- - -
	1 0 0	- - -
	1 0	- - -
	1	- - -
$F_7 - 1 \text{ zeros}$	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0 0	- - -
	1 0 0 0 0 0 0	- - -
	1 0 0 0 0 0	- - -
	1 0 0 0 0	- - -
	1 0 0 0	- - -
	1 0 0	- - -
	1 0	- - -
	1	- - -
$F_8 - 1 \text{ zeros}$	.....	and so on

**Fig. La Scala di Fibonacci.** The staircase structure of incidence matrix  $\zeta \in I(P, R)$  = incidence algebra of  $P$  over the commutative ring  $R$ . See [12](2003)

For many pictures and further progress on cobweb posets and KoDAGs consult [17-29,31-42].

The characteristic function  $\chi_F(\leq) = \zeta_F(\leq)$  of the partial order relation  $\leq$  of the graded  $F$ -cobweb poset  $P$  - or in brief just  $\zeta$  - might be expressed by Kronecker delta as here down ([18] (2003) and see also [12,40,18,16]) **for any natural numbers valued sequence  $F$**  - including the motivating example with Fibonacci sequence  $F$  denominated graded cobweb poset  $P$  for which it reads ( $F_0 = \mathbf{0}$ ):

### Kwaśniewski $\zeta$ 2003 formula

$$\zeta = \zeta_1 - \zeta_0$$

where for  $x, y \in \mathbf{N}$ :

$$\begin{aligned} \zeta_1(x, y) &= \sum_{k=0}^{\infty} \delta(x + k, y) \\ \zeta_0(x, y) &= \sum_{k \geq 0} \sum_{s \geq 0} \delta(x, F_{s+1} + k) \sum_{r=1}^{F_s - k - 1} \delta(k + F_{s+1} + r, y) \end{aligned}$$

and naturally

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

Right from the Hasse diagram of  $P$  here now obvious observations follow.

**Observation 4.1.** *The number of maximal chains starting from the root (level  $F_1$ ) to reach any point at the  $n$ -th level labeled by  $F_n$  is equal to  $n_F!$ .*

**Observation 4.2.** *The number of maximal chains starting from any fixed point at the level labeled by  $F_k$  to reach any point at the  $n$ -th level labeled by  $F_n$  is equal to  $n_F^m$  ( $n = k + m$ ) .*

Let us denote by  $P_m$  a subposet of  $P$  with vertices up to  $m$ -th level vertices

$$\bigcup_{s=1}^m \Phi_s ; \Phi_s \text{ is the set of elements of the } s-\text{th level}$$

**Attention please.**  $P_m$  depending on the context is also to be viewed on as representing the set of maximal chains. It is naturally a particular case of the layer of graded poset called cobweb poset [1-23] - for the reader convenience - see **Appendix** with basic ponderables. For the sake of coming right now combinatorial interpretation, all layers on one hand are viewed on as subposets while on the other hand are also to be viewed on as representing the set of maximal chains.

Consider now the following behavior of a "sub-cob" moving from any given point of the  $F_k$  level of the poset up. It behaves as it has been born right there and can reach at first  $F_2$  points up then  $F_3$  points up,  $F_4$  and so on - thus climbing up to the level  $F_{k+m} = F_n$  of the poset  $P$ . It can see - as its Great Ancestor at the root  $F_1$ -th level and potentially follow one of its own accessible finite subposet  $P_m$ . One of many  $P_m$ 's rooted at the  $k$ -th level might be found. How many?

The answer for the **Fibonacci** sequence  $F$  was given in [12] (**2003**) :

**Observation 4.3.** Let  $n = k + m$ . The number of subposets equipotent to subposet  $P_m$  rooted at any **fixed** point at the level labeled by  $F_k$  and ending at the  $n$ -th level labeled by  $F_n$  is equal to

$$\binom{n}{m}_F = \binom{n}{k}_F = \frac{n_F^k}{k_F!}.$$

Equivalently - now in a bit more mature **2009** year the answer is given simultaneously viewing layers as biunivoquely representing maximal chains sets. Let us make it formal.

Such recent equivalent formulation of this combinatorial interpretation is to be found in [20] from where we quote it here down.

Let  $\{F_n\}_{n \geq 0}$  be a natural numbers valued sequence with  $F_0 = 1$  (or  $F_0! \equiv 0!$  being exceptional as in case of Fibonacci numbers). Any such sequence uniquely designates both  $F$ -nomial coefficients of an  $F$ -extended umbral calculus as well as  $F$ -cobweb poset introduced by this author (see :the source [19] from 2005 and earlier references therein). If these  $F$ -nomial coefficients are natural numbers or zero then we call the sequence  $F$  - the  **$F$ -cobweb admissible sequence**.

**Definition 4.1.** Let any  $F$ -cobweb admissible sequence be given then  $F$ -

nomial coefficients are defined as follows

$$\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)_F!} = \frac{n_F \cdot (n-1)_F \cdot \dots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \dots \cdot k_F} = \frac{n_F^k}{k_F!}$$

while  $n, k \in N$  and  $0_F! = n_F^0 = 1$ .

**Definition 4.2.**  $C_{max}(P_n) \equiv \{c = \langle x_0, x_1, \dots, x_n \rangle, x_s \in \Phi_s, s = 0, \dots, n\}$  i.e.  $C_{max}(P_n)$  is the set of all maximal chains of  $P_n$

**Definition 4.3.** Let

$$C_{max}\langle \Phi_k \rightarrow \Phi_n \rangle \equiv \{c = \langle x_k, x_{k+1}, \dots, x_n \rangle, x_s \in \Phi_s, s = k, \dots, n\}.$$

Then the  $C\langle \Phi_k \rightarrow \Phi_n \rangle$  set of Hasse sub-diagram corresponding maximal chains defines biunivoquely the layer  $\langle \Phi_k \rightarrow \Phi_n \rangle = \bigcup_{s=k}^n \Phi_s$  as the set of maximal chains' nodes and vice versa - for these **graded** DAGs (KoDAGs included).

The **equivalent** to that of of **Obsevation 3** formulation of combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following.

**Theorem [20]**

(Kwaśniewski) For  $F$ -cobweb admissible sequences  $F$ -nomial coefficient  $\binom{n}{k}_F$  is the cardinality of the family of equipotent to  $C_{max}(P_m)$  mutually disjoint maximal chains sets, all together **partitioning** the set of maximal chains  $C_{max}\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$  of the layer  $\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$ , where  $m = n - k$ .

For February 2009 readings on further progress in combinatorial interpretation and application of the presented author invention i.e. partial order posets named cobweb posets and their's corresponding encoding Hasse diagrams KoDAGs see [19-29,31-42] and references therein. For active presentation of cobweb posets see [40]. This electronic paper is an update of [41]

## 5 Summarizing , Concluding ad upgrade and ad references remarks

**Remark 1** The matrix elements of  $\zeta(x, y)$  were given in 2003 ([18] Kwaśniewski) using  $x, y \in N \cup \{0\}$  labels of vertices in their "natural" linear order:

1. set  $k = 0$ ,
2. then label subsequent vertices - from the left to the right - along the level  $k$ ,
3. repeat 2. for  $k \rightarrow k + 1$  until  $k = n + 1$  ;  $n \in N \cup \{\infty\}$

As the result we obtain the  $\zeta$  matrix for Fibonacci sequence as presented by the Fig. *La Scala di Fibonacci* dating back to 2003 [18,41].

Inspired [8,9,13] by Gauss  $n_q = q^0 + q^1 + \dots + q^{n-1}$  finite geometries numbers and in the spirit of Knuth "notationlogy" [30] we shall refer here also to the upside down notation effectiveness as in [21-24,20] or earlier in [8,9,10,13], (consult also the Appendix in [33]). **As for** that upside down attitude  $F_n \equiv n_F$  being much more than "just a convention" to be used substantially in what follows as well as for the sake of completeness - **let us** quote it as The Principle according to Kwasniewski [42] where it has been formulated as an "of course" Principle i.e. simultaneously trivial and powerful statement.

### The Upside Down Notation Principle.

1. **Let** the statement  $s(F)$  depends only on the fact that  $F$  is a natural numbers valued statement.
2. **Then** if one proves that  $s(N) \equiv s(\langle n \rangle_{n \in N})$  is true - the statement  $s(F) \equiv s(\langle n_F \rangle_{n \in N})$  is also true. Formally - use equivalence relation classes induced by co-images of  $s : \{F\} \mapsto 2^{\{1\}}$  and proceed in a standard way.

In order to proceed further let us now recall-rewrite purposely here Kwaśniewski 2003 - formula for  $\zeta$  function of **arbitrary** cobweb poset so as to see that its' algorithm rules authomatically make it valid for all  $F$ -cobweb posets where  $F$  is any natural numbers valued sequence i.e. with  $F_0 > 0$ .  $I(\Pi, R)$  stays for the incidence algebra of the poset  $\Pi$  over the commutative ring  $R$ .

$$\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y)$$

$$\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y)$$

$$\zeta_0(x, y) = \sum_{k \geq 1} \sum_{s \geq 0} \delta(x, F_{s+1} + k) \sum_{r=1}^{F_s - k - 1} \delta(k + F_{s+1} + r, y)$$

and naturally

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

The above formula for  $\zeta \in I(\Pi, R)$  rewritten in ( $F_s \equiv s_F$ ) upside down notation equivalent form as below is of course **valid for all** cobweb posets.

$$\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y),$$

$$\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y),$$

$$\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r=1}^{(s-1)_F - k - 1} \delta(x + r, y).$$

**Note.**  $+\zeta_1$  "produces the Pacific ocean of 1's" in the whole upper triangle part of a would be incidence algebra  $\sigma \in I(\Pi, R)$  matrix elements

**Note.**  $-\zeta_0$  cuts out 0's i.e. thus producing "zeros'  $F$ -La Scala staircase", in the 1's delivered by  $+\zeta_1$ .

This results exactly in forming 0's rectangular triangles:  $s_F - 1$  of them at the start of subsequent stair and then down to one 0 till - after  $s_F - 1$  rows passed by one reaches a half-line of 1's which is running to the right-right to infinity and thus marks the next in order stair of the  $F$ - La Scala.

The  $\zeta$  matrix explicit formula was given for arbitrary graded posets with the finite set of minimal in terms of natural join of bipartite digraphs in SNACK

= the Sylvester Night Article on KoDAGs and Cobwebs = [21].

What was said is equivalent to the fact that the cobweb poset coding La Scala is of the natural join operation origin while thus producing  $\zeta$  matrix [23,21,24] which is of the form (quote from SNACK = [21], see: Subsection 2.6.)

The explicit expression for zeta matrix  $\zeta_F$  of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued  $F$ - sequence was given in (here) [23] due to more than mnemonic efficiency of the up-side-down notation being applied (see [23] and references therein). With this notation inspired by Gauss and replacing  $k$  - natural numbers with "  $k_F$ " numbers one gets

$$\mathbf{A}_F = \begin{bmatrix} 0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\ 0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\ 0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\ 0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\ \dots & etc & \dots & \text{and so on} & \dots \end{bmatrix}$$

and

$$\begin{aligned} \zeta_F &= \exp_{\odot}[\mathbf{A}_F] \equiv (1 - \mathbf{A}_F)^{-1 \odot} \equiv I_{\infty \times \infty} + \mathbf{A}_F + \mathbf{A}_F^{\odot 2} + \dots = \\ &= \begin{bmatrix} \mathbf{I}_{1_F \times 1_F} & I(1_F \times \infty) \\ O_{2_F \times 1_F} & \mathbf{I}_{2_F \times 2_F} & I(2_F \times \infty) \\ O_{3_F \times 1_F} & O_{3_F \times 2_F} & \mathbf{I}_{3_F \times 3_F} & I(3_F \times \infty) \\ O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & \mathbf{I}_{4_F \times 4_F} & I(4_F \times \infty) \\ \dots & etc & \dots & \text{and so on} & \dots \end{bmatrix} \end{aligned}$$

where  $I(s \times k)$  stays for  $(s \times k)$  matrix of ones i.e.  $[I(s \times k)]_{ij} = 1$ ;  $1 \leq i \leq s, 1 \leq j \leq k$ . and  $n \in N \cup \{\infty\}$

In the  $\zeta_F$  formula from [23]  $\odot$  denotes the Boolean product, hence - used for Boolean powers too. We readily recognize from its block structure that  $F$ -La Scala is formed by **upper zeros** of block-diagonal matrices  $\mathbf{I}_{k_F \times k_F}$  which

sacrifice these their **zeros** to constitute the  $k$ -th subsequent stair in the  $F$ -La Scala descending and descending far away down to infinity. Thus the cobweb poset coding La Scala is due to the natural join origin of  $\zeta$  matrix.

Note now that because of  $\delta$ 's under summations in the former  $\zeta$  formula the following is obvious:

$$\begin{aligned} 1 \leq r = y - x &\leq (s-1)_F - k - 1 \equiv 1 \leq r = y - k - s_F \leq s-1)_F - k - 1 \equiv \\ &\equiv 1 \leq r = y \leq s_F - (s-1)_F - 1. \end{aligned}$$

Because of that the above last expression of the  $\zeta$  expressed in terms of  $\delta \in I(\Pi, R)$  may be still simplified [for the sake of verification and portraying via computer simple program implementation]. Namely the following is true:

$$\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y),$$

where

$$\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y),$$

[ note:  $+\zeta_1$  "produces the Pacific ocean of 1's" in the whole upper triangle part of a would be incidence algebra  $\sigma \in I(\Pi, R)$  matrix elements],

and where

$$\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r \geq 1}^{s_F + (s-1)_F - 1} \delta(r, y),$$

[ note then again that  $-\zeta_0$  cuts out "one's  $F$ -La Scala staircase" in the 1's provided by  $+\zeta_1$  ].

Note, that for  $F = \text{Fibonacci}$  this still more simplifies as then

$$s_F + (s-1)_F - 1 = (s+1)_F.$$

**Remark 1.1.** ad Knuth notation [30] indicated back to me by Maćiej Dziemiańczuk

In the wise "‘notationlogy’" note [30] among others one finds notation just for the purpose here (Maciej Dziemiańczuk’s observation)

$$[s] = \begin{cases} 1 & \text{if } s \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Using this makes my last above expression of the  $\zeta$  in terms of  $\delta$  still more transparent and handy if rewritten in Donald Ervin Knuth’s notation [30]. Namely:

$$\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y)$$

$$\zeta_1(x, y) = [x \leq y]$$

$$\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} [x = k + s_F][1 \leq y \leq s_F + (s-1)_F - 1].$$

Note, that for  $F = \text{Fibonacci}$  this still more simplifies as then

$$s_F + (s-1)_F - 1 = (s+1)_F.$$

### Remark 1.2. Knuth notation [30] - and Dziemiańczuk’s guess

It was remarked by my Gdańsk University Student Maciej Dziemiańczuk - that my  $\zeta \in I(\Pi, R)$  (equivalent) expressions are valid according to him only for  $F = \text{Fibonacci}$  sequence. In view of the Upside Down Notation Principle if any of these is valid for any particular natural numbers valued sequence  $F$  it should be true for all of the kind.

His being doubtful led him to creative invention of his own.

Here comes the formula postulated by him.

$$\zeta(x, y) = [x \leq y] - [x < y] \sum_{n \geq 0} [(x > S(n))[y \leq S(n+1)]],$$

where

$$S(n) = \sum_{k \geq 1}^n k_F$$

**Exercise.** My reply to this guess is the following Exercise.

Let  $x, y \in N \cup \{0\}$  be the labels of vertices in their "natural" linear order as explained earlier.

Prove the claim:

*Dziemiańczuk formula is equivalent to Kwaśniewski formulas.*

- What is against?
- What is for? My "for" is the Socratic Method question. Why not use the arguments in favor of

$$\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r \geq 1}^{s_F + (s-1)_F - 1} \delta(r, y),$$

Hint. Use the same argumentation.

Here are some illustrative examples-exercises with pictures [Figures 2,3,4] delivered by Maciej Dziemiańczuk's computer personal service using the above Dziemiańczuk formula.

**Remark 2. Krot Choice.** While the above is established it is a matter of simple observation by inspection to find out how does the Möbius matrix  $\mu = \zeta^{-1}$  looks like. Using in [25,26] this author example and expression for  $\zeta$  matrix this has been accomplished first (see also [26]) for Fibonacci sequence and then - via automatic extension - the same formula was given for  $F$  sequences as above by my former student in her recent articles [27,28]. Here is her formula for the cobweb posets' Möbius function (see: (6) in [28]).

$$\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) - \delta(t+1, v) + \sum_{k=2}^{\infty} \delta(t+k, v)(-1)^k \prod_{i=t+1}^{v-1} F_i - 1$$

Now bearing in mind the Upside Down Notation Principle **for all**  $F$ -cobweb posets with  $F_0 > 0$  (as it should be for natural numbers valued sequences) we may now rewrite the above in coordinate grid  $Z \times Z$  as below.

Let  $x = \langle s, t \rangle$  and  $y = \langle u, v \rangle$  where  $1 \leq s \leq F_t$ ,  $1 \leq u \leq F_v$  while  $t, v \in N$ . Then

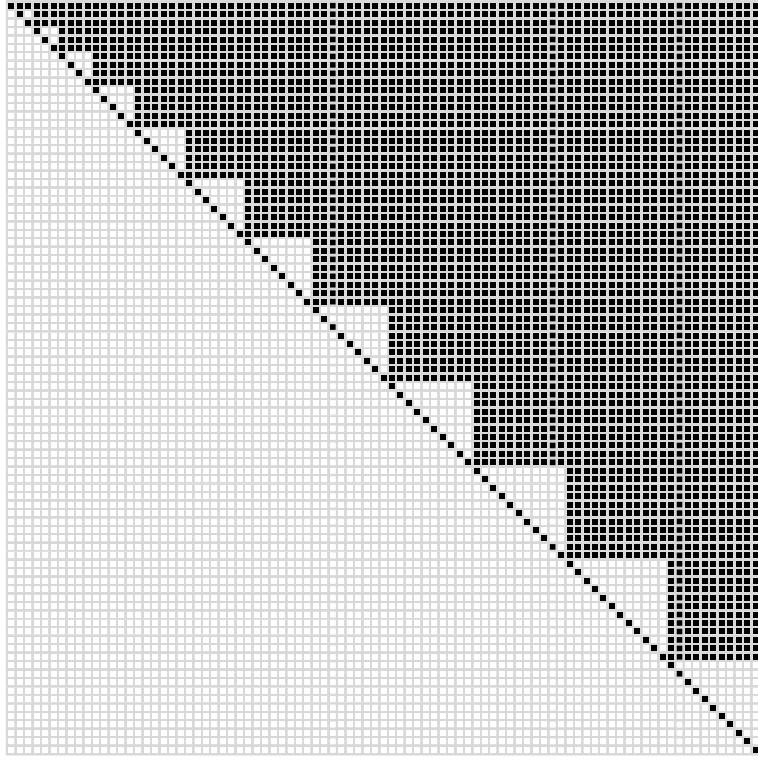


Figure 2: Display of the  $\zeta = 90 \times 90$ ). The subposet  $P_t$  of the  $N$  i.e. integers sequence  $N$ -cobweb poset.  $t = ?$

$$\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t+k, v)(-1)^k \prod_{i=t+1}^{v-1} (i-1)_F$$

The further relevant references of the same author see [27-29]. The above rewritten Möbius function formula is of course literally **valid for all** natural numbers valued sequences  $F$ . Consult: the recent note "On Characteristic Polynomials of the Family of Cobweb Posets" [29].

The author of [25] introduces parallelly also another form of  $\zeta$  function formula and since now on -except for [27]- in subsequent papers [26,28,29] their author uses the formula for  $\zeta$  function in this another form. Namely - this other form formula for  $\zeta$  function in the present authors' grid coordinate

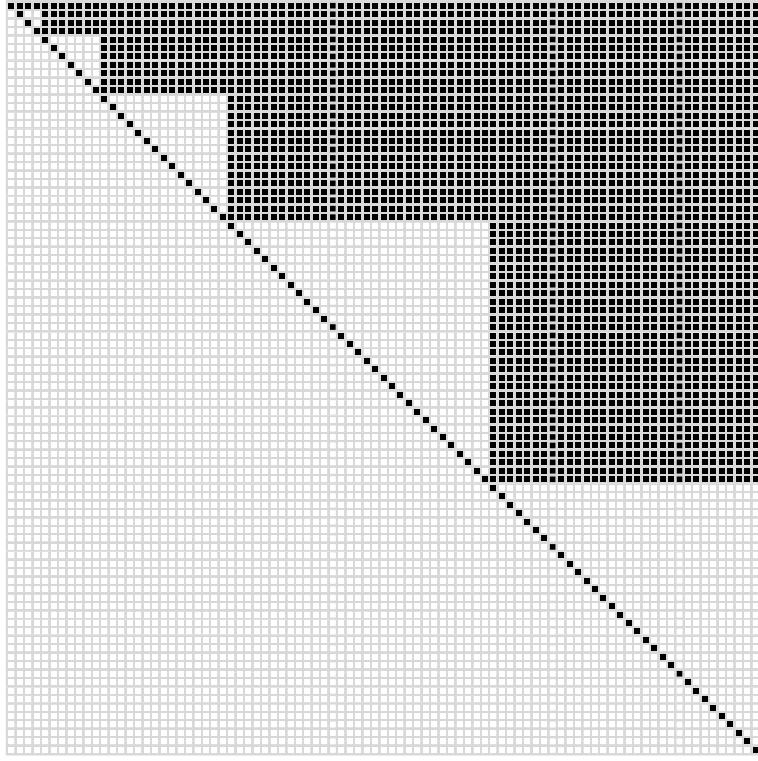


Figure 3: Display of the  $\zeta = 90 \times 90$ ). The subposet  $P_t$  of the  $F = \text{Gaussian integers sequence}$  ( $q = 2$ ).  $F$ -cobweb poset. $t = ?$

system description of the cobweb posets was given by Krot in her note on Möbius function and Möbius inversion formula for Fibonacci cobweb poset [23] with  $F$  designating the Fibonacci cobweb posets. In [24] the formula for the Möbius function for Fibonacci sequence  $F$  was rightly treated as literally **valid for all** natural numbers valued sequences  $F$  and the alternative. Of course the same concerns the  $\zeta$  function formulas - the former and the latter. Here is this other latter form formula for  $\zeta$  function formula (see: (7) in [25] or (1) in [27]).

Let  $x = \langle s, t \rangle$  and  $y = \langle u, v \rangle$  where  $1 \leq s \leq F_t$ ,  $1 \leq u \leq F_v$  while  $t, v \in N$ . Then

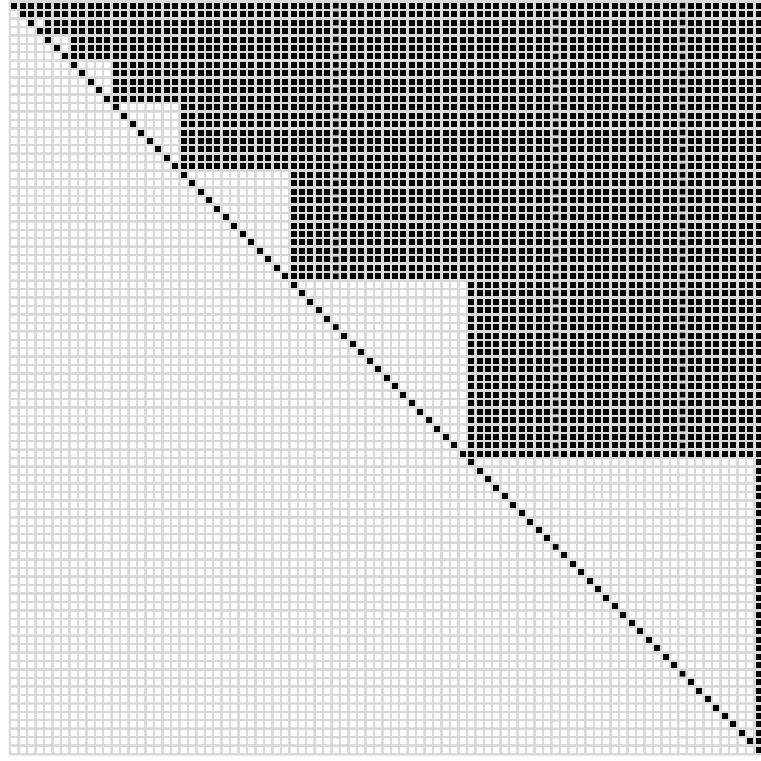


Figure 4: Display of the  $\zeta = 90 \times 90$ ). The subposet  $P_t$  of the  $F =$  Fibonacci-cobweb poset. $t = ?$

$$\zeta(x, y) = \zeta(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t+k, v)$$

where here - recall  $(a, b \in Z)$ :

$$\delta(a, b) = \begin{cases} 1 & \text{for } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

**Farewell interactive question.** *Are then all the presented  $\zeta$  incidence function matrix of  $F$ -denominated cobweb posets equivalent?*

## 6 Appendix [42]

**Cobweb posets and KoDAGs' ponderables of the authors relevant productions.**

**Definition 6.1.** Let  $n \in N \cup \{0\} \cup \{\infty\}$ . Let  $r, s \in N \cup \{0\}$ . Let  $\Pi_n$  be the graded partial ordered set (poset) i.e.  $\Pi_n = (\Phi_n, \leq) = (\bigcup_{k=0}^n \Phi_k, \leq)$  and  $\langle \Phi_k \rangle_{k=0}^n$  constitutes ordered partition of  $\Pi_n$ . A graded poset  $\Pi_n$  with finite set of minimal elements is called **cobweb poset** iff

$$\forall x, y \in \Phi \text{ i.e. } x \in \Phi_r \text{ and } y \in \Phi_s \text{ } r \neq s \Rightarrow x \leq y \text{ or } y \leq x,$$

$$\Pi_\infty \equiv \Pi.$$

**Note.** By definition of  $\Pi$  being graded its levels  $\Phi_r \in \{\Phi_k\}_k^\infty$  are independence sets and of course partial order  $\leq$  up there in Definition 6.1. might be replaced by  $<$ .

The Definition 6.1. is the reason for calling Hasse digraph  $D = \langle \Phi, \leq \cdot \rangle$  of the poset  $(\Phi, \leq)$  a **KoDAG** as in Professor Kazimierz Kuratowski native language one word **Komplet** means **complete ensemble** - see more in [23] and for the history of this name see: The Internet Gian-Carlo Polish Seminar **Subject 1. oDAGs and KoDAGs in Company** (Dec. 2008).

**Definition 6.2.** Let  $F = \langle k_F \rangle_{k=0}^n$  be an arbitrary natural numbers valued sequence, where  $n \in N \cup \{0\} \cup \{\infty\}$ . We say that the cobweb poset  $\Pi = (\Phi, \leq)$  is **denominated** (encoded=labelled) by  $F$  iff  $|\Phi_k| = k_F$  for  $k = 0, 1, \dots, n$ .

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